Using Tensor Diagrams to Represent and Solve Geometric Problems

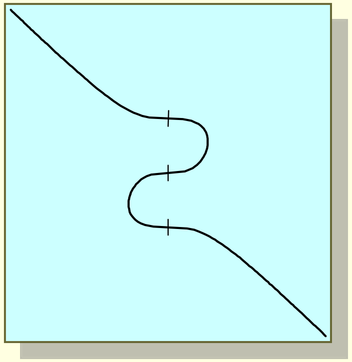
Introduction

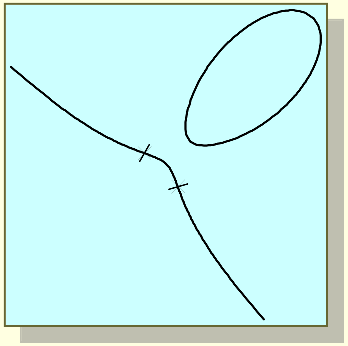
A Geometry Problem
Cubic Curves

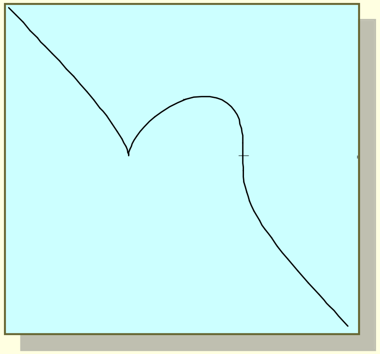
$$f(X,Y) = AX^3 + 3BX^2Y + 3CXY^2 + DY^3$$
$$+3EX^2 + 6FXY + 3GY^2$$
$$+3HX + 3JY$$
$$+K = 0$$

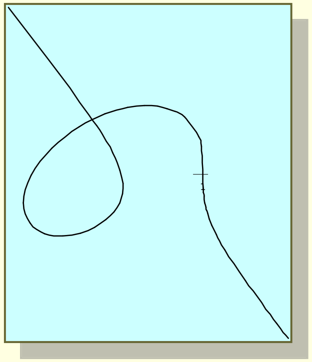
Possible Cubic Curve Shapes

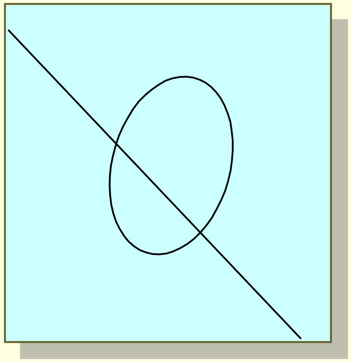
$$
\begin{aligned}
f(X,Y) = AX^3 + 3BX^2Y + 3CXY^2 + DY^3 \\
+3EX^2 + 6FXY + 3GY^2 \\
+3HX + 3JY \\
+K = 0
\end{aligned}
$$







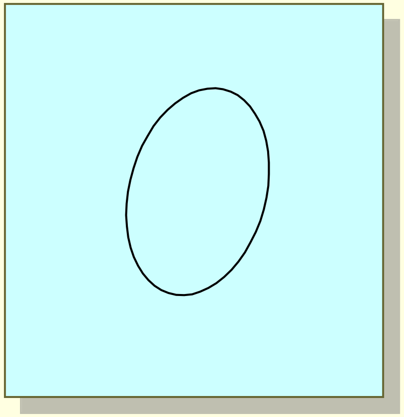


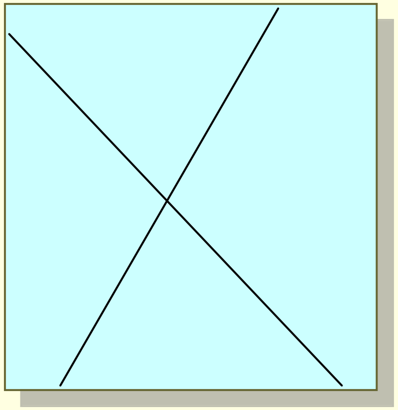


$\mathbf{D}(A, B, C, \ldots, J, K)$

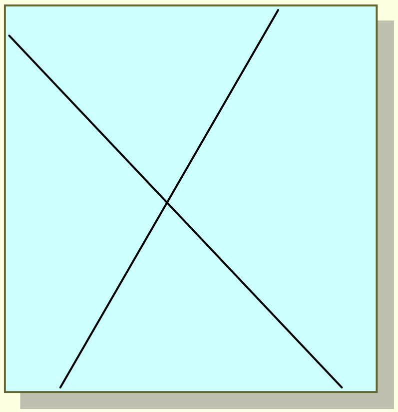
Quadratic Curves

$$f(X,Y) = AX^2 + 2BXY + CY^2$$
$$+ 2DX + 2EY$$
$$+ F = 0$$





Discriminant of Quadratic



$$\mathbf{D}\left(A, B, C, D, E\right) = 0$$

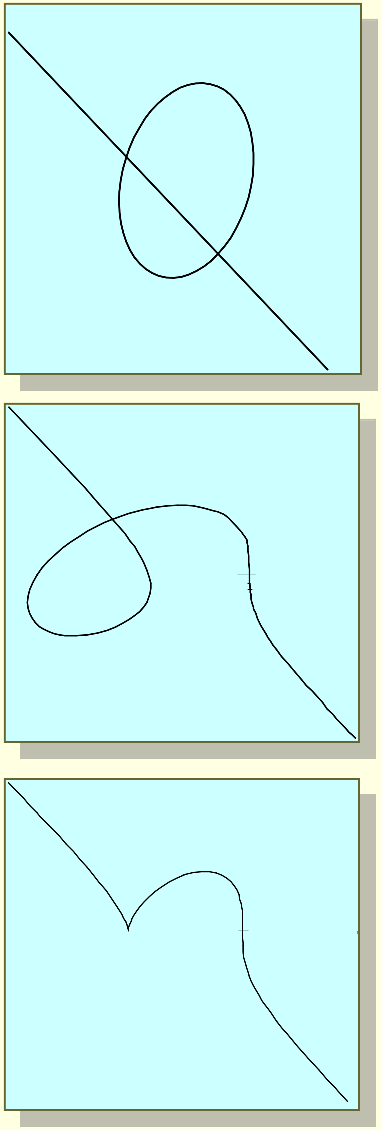
$$ACF + 2BED - D^2C - E^2A - B^2F = 0$$

D is degree 3 in A…E

D has 5 terms

$$\det\begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} = 0$$

Discriminant of Cubic



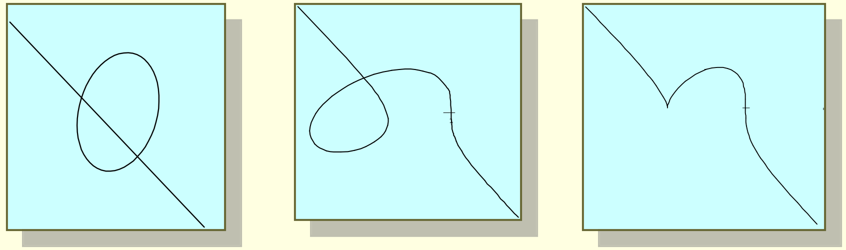
$$\mathbf{D}\left(A,B,C,D,E,F,G,H,J,K\right)=0$$

G. Salmon (1879):

D is degree 12 in *A*…*K*

D has over 10,000 terms

Discriminant of Cubic

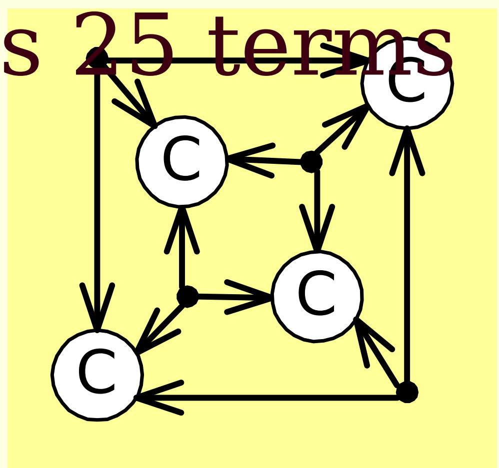


$$\mathbf{D} = 64S^3 + T^2$$

S: degree 4 in $A \ldots K$

has 25 terms

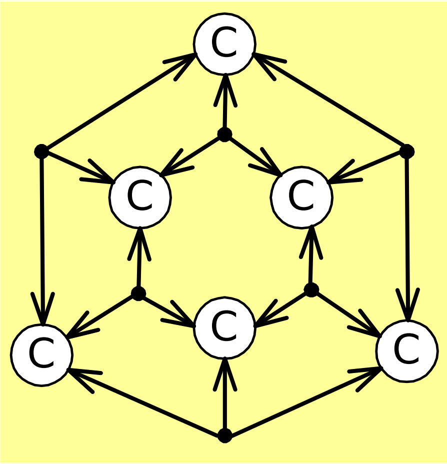
$$s = -\frac{1}{24}$$



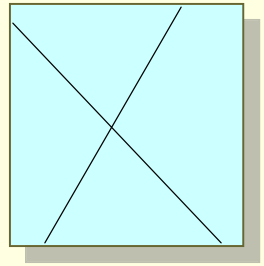
T: degree 6 in $A \ldots K$

has

$$T = -\frac{1}{6}$$



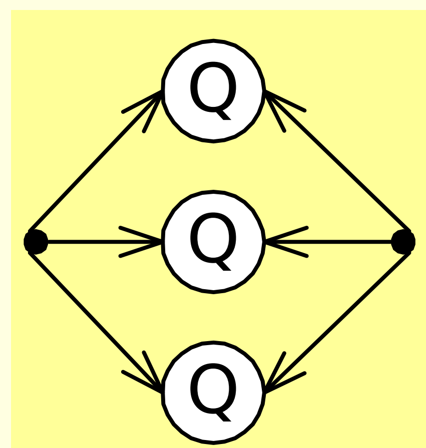
Discriminant of Quadratic



$$\Delta\left(A,B,C,D,E\right)=0$$

$$ACF+2BED-D^2C-E^2A-B^2F=0$$

$$\det\begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}=0$$



$=0$

Tensor Diagrams

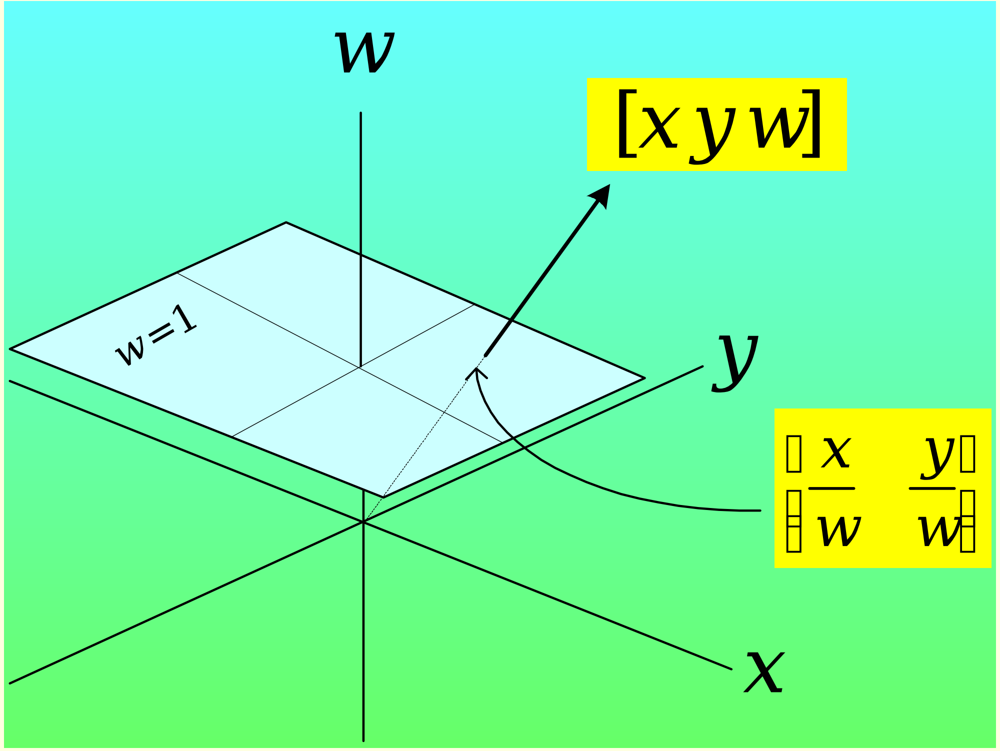
- Express complicated polynomials visually
- Aid in manipulation of polynomials
- Derive existing results easily
- Derive new results (?)

Homogeneous Geometry

$$P = \begin{bmatrix} X & Y \end{bmatrix}$$

$$P = \begin{bmatrix} x & y & w \end{bmatrix}$$

$$aP = \begin{bmatrix} ax & ay & aw \end{bmatrix}$$



The Homogeneous Universes

1D (Polynomials)

$$f(X) = AX^2 + BX + C$$

$$f(x,w) = Ax^2 + Bxw + Cw^2$$

2D (Curves)

$$f(X,Y) = DX^2 + EY + F$$

$$f(x,y,w) = Dx^2 + Eyw + Fw^2$$

3D (Surfaces)

$$f(X,Y,Z) = GX^2 + HY + JZ$$

$$f(x,y,z,w) = Gx^2 + Hyw + Jzw$$

The Homogeneous Universes

	Euclidean	Projective
Polynomials	1D: $[X]$	1DH: $[x\ w]$
Curves	2D: $[X\ Y]$	2DH: $[x\ y\ w]$
Surfaces	3D: $[X\ Y\ Z]$	3DH: $[x\ y\ z\ w]$

The Matrix of Knowledge

	1DH	2DH	3DH
Linear			
Quadratic			
Pairs of quadratics			
Cubic			
Quadratic and Cubic			
Pair of Cubics			
Quartic			

Tensor Diagrams

- A Work in Progress
  - Some simple results not complete
  - Lot of stuff is still rough around the edges
  - Tutorial notes are obsolete
- Want to show what I’ve figured out so far
- Enlist others in finding more results